Model: $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

Weak formulation: Find $u \in H^1_0(\Omega)$ s.t.

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \qquad \forall v \in H^1_0(\Omega)$$

- Well posedness: relies on Poincaré ineq.

① on $H^1_0(\Omega)$: $\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2$

coercivity: $\|u\|_{H^1}^2 \underset{\substack{\text{Poincaré} \\ \text{in } H^1_0}}{\leq} C\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2$

$\leq (C+1)\, a(u,u)$

② on $H^1_0(\Omega)$: $\|u\|_{H^1_0(\Omega)}^2 = \|\nabla u\|_{L^2(\Omega)}^2 = a(u,u)$

This is a norm because of Poincaré.

- Then: $v = u$ in formulation:

$$\|\nabla u\|_{L^2}^2 = \int_\Omega f u \leq \|f\|_{L^2} \|u\|_{L^2}$$

$$\underset{\substack{\text{Poincaré} \\ (\text{in } H^1_0)}}{\leq} C\|f\|_{L^2} \|\nabla u\|_{L^2}$$

so $\boxed{\|\nabla u\|_{L^2} \leq C\|f\|_{L^2}}$

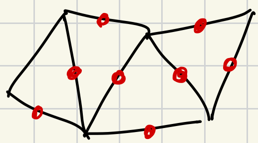
* <u>Crouzeix-Raviart</u>: non-conforming FE space.

- $\mathcal{T}_h$: conforming simplicial mesh of Ω

- $V_h = \{ v_h \in L^2(\Omega) : (v_h)_{|T} = v_T \in \mathbb{P}^1(T) \ \forall T \in \mathcal{T}_h,$

  v_h continuous at face centers $\}$

$\not\subset H^1_0$



<u>Remark</u>: $\forall F$ between T, T'

$$\int_F (v_h)_{|T} = \int_F (v_h)_{|T'}$$

$$\Leftrightarrow \quad (v_h)_{|T}(\bar{x}_F) = (v_h)_{|T'}(\bar{x}_F)$$

where $\bar{x}_F = \frac{1}{|F|} \int_F x$.

$$V_{h,0} = \{ v_h \in V_h : \int_F v_h = 0 \quad \forall F \in \mathcal{F}_h^b \}$$

($\mathcal{F}_h$ = faces of mesh, $\mathcal{F}_h^b$ = faces lying on $\partial\Omega$)

- CR-scheme for Poisson problem: Find $u_h \in V_{h,0}$

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v_h = \int_\Omega f v_h \quad \forall v_h \in V_{h,0}$$

broken gradient: $(\nabla_h u_h)_{|T} = \nabla (u_h)_{|T}$

- Well-posedness: we need Poincaré in $V_{h,0}$!

Lemma: There is $C>0$ s.t.

$$\|v_h\|_{L^2} \le C \|\nabla_h v_h\|_{L^2} \quad \forall v_h \in V_{h,0}$$

Proof: $v_h \in L^2(\Omega)$ so there is $\sigma \in H^1(\Omega)^d$ s.t. $\operatorname{div} \sigma = v_h$ and $\|\sigma\|_{H^1_0} \le C \|v_h\|_{L^2}$

Then

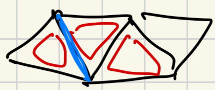
$$\|v_h\|_{L^2}^2 = \int_\Omega v_h \cdot v_h = \int_\Omega v_h \operatorname{div} \sigma$$

$$= \sum_T \int_T v_T \operatorname{div} \sigma$$

$$\overset{IBP}{=} \sum_T \left(-\int_T \nabla v_T \cdot \sigma + \int_{\partial T} v_T \, \sigma \cdot n_T \right)$$

$$= -\int_\Omega \nabla_h v_h \cdot \sigma + T_2$$

We have: $T_2 = \sum_T \int_{\partial T} v_T \, \sigma \cdot n_T$



$$= \sum_{F \in \mathcal{F}_h^i} \left(\int_F v_T\, \sigma \cdot n_T + \int_F v_{T'}\, \sigma \cdot n_{T'} \right)$$

$(\mathcal{F}_h^i = \mathcal{F}_h \setminus \mathcal{F}_h^b)$

$$+ \sum_{F \in \mathcal{F}_h^b} \int_F v_T\, \sigma \cdot n_T$$

where: if $F \in \mathcal{F}_h^i$: T, T' simplices on each side of F

if $F \in \mathcal{F}_h^b$: T simplex that touches F

$$= \sum_{F \in \mathcal{F}_h^i} \left(\int_F (v_T - v_F)\, \sigma \cdot n_T + \int_F (v_{T'} - v_F)\, \sigma \cdot n_{T'} \right)$$

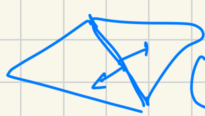
$$+ \sum_{F \in \mathcal{F}_h^b} \int_F (v_T - v_F)\, \sigma \cdot n_T$$

where $v_F = \frac{1}{|F|} \int_F v_h \quad \left(= \int_F (v_h)_T = \int_F (v_h)_{T'} \right)$

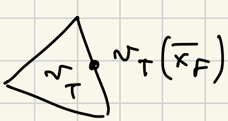
$= (v_h)(\bar{x}_F) \quad (= 0 \text{ if } F \in \mathcal{F}_h^b)$

Why: because

$$0 = \sum_{F \in \mathcal{F}_h^i} \int_F v_F \underbrace{(\sigma \cdot n_T + \sigma \cdot n_{T'})}_{} + \sum_{F \in \mathcal{F}_h^b} \int_F \underset{\overset{\|}{0}}{v_F}\, \sigma \cdot n_T$$



$(n_T)_F + (n_{T'})_F = 0$



$$|v_T - v_T(\bar{x}_F)| \lesssim h_T\, |\nabla v_T|$$

so $$\|v_T - v_F\|_{L^2(F)} \lesssim h_T\, \|\nabla v_T\|_{L^2(F)}$$

$$|T_2| \leq \sum_{F \in \mathcal{F}_h^i} \left(\|\sigma\|_{L^2(F)} \|v_T - v_F\|_{L^2(F)} + \dots T' \right)$$

$$+ \sum_{F \in \mathcal{F}_h^b}$$

$$\leq \sum_{F \in \mathcal{F}_h^i} \underbrace{\|\sigma\|_{L^2(F)}}_{\text{continuous}} h_T^{\frac{1}{2}+\frac{1}{2}} \underbrace{\|\nabla v_T\|_{L^2(F)}}_{\text{discrete}} + \dots$$

$$+ \dots$$

Trace inequalities:

① continuous : $$h_T^{1/2} \|w\|_{L^2(F)} \leq C\|w\|_{L^2(T)} + C h_T \|\nabla w\|_{L^2(T)} \quad \forall w \in H^1(T)$$

② discrete : $$h_T^{1/2} \|q\|_{L^2(F)} \leq C \|q\|_{L^2(T)} \quad \forall q \in \mathbb{P}^\ell(T)$$

$$T_2 \leq C \sum_{F \in \mathcal{F}_h^i} \left(\|\sigma\|_{L^2(T)} + h_T \|\nabla \sigma\|_{L^2(T)} \right) \|\nabla v_T\|_{L^2(T)}$$

$$+ \dots T'$$

$$+ C \sum_{\mathcal{F}_h^b}$$

$$\leq C \sum_{T \in \mathcal{T}_h} \left(\|\sigma\|_{L^2(T)} + h_T \|\nabla \sigma\|_{L^2(T)} \right) \|\nabla v_T\|_{L^2(T)}$$

$$\underset{CS}{\leq} \left(\|\sigma\|^2_{L^2(\Omega)} + h \|\nabla \sigma\|^2_{L^2(\Omega)} \right)^{1/2} \|\nabla_h v_h\|_{L^2(\Omega)}$$

So $$\|v_h\|_{L^2}^2 \leq C \underbrace{\|z\|_{H^1(\Omega)}}_{\leq C \|v_h\|_{L^2(\Omega)}} \|\nabla_h v_h\|_{L^2(\Omega)}$$

so $$\|v_h\|_{L^2} \leq C \|\nabla_h v_h\|_{L^2(\Omega)}$$ $\square$

* Shifting point of view:

- Start from DOF:

$$v_h \in V_{h,0} \xrightarrow{DOF} (v_F)_{F \in \mathcal{F}_h}, \quad v_F = \frac{1}{|F|}\int_F v_h$$

- Reconstruct v_T from $(v_F)_{F \in \mathcal{F}_T}$

($\mathcal{F}_T$: faces of T)

(1) Take $w \in \mathbb{P}^1(T)$:

$$\int_T \nabla v_T \cdot \nabla w = -\int_T v_T \cancel{\Delta w}^{\,0} + \sum_{F \in \mathcal{F}_T} \int_F v_T \nabla w \cdot n_{TF}$$

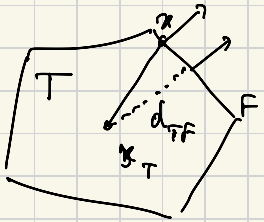
$$= \sum_{F \in \mathcal{F}_T} \underbrace{\left(\int_F v_T\right)}_{\int_F v_F} \nabla w \cdot n_{TF}$$

$$= \sum_{F \in \mathcal{F}_T} \int_F v_F \nabla w \cdot n_{TF}$$

Rule $\quad (v_F)_{F\in\mathcal{F}_T} \longrightarrow \nabla v_T$

(2) $$\int_T v_T = \frac{1}{d}\int_T v_T \,\mathrm{div}(x-\bar{x}_T)$$

$$= \frac{1}{d}\sum_{F\in\mathcal{F}_T}\int_F v_T \underbrace{(x-\bar{x}_T)\cdot n_{TF}}_{\substack{d_{TF} \\ = \mathrm{dist}(\bar{x}_T, F)}} - \frac{1}{d}\underbrace{\int_T \nabla v_T\cdot(x-\bar{x}_T)}_{\nabla v_T\cdot\underbrace{\int_T(x-\bar{x}_T)}_{0}}$$



$$d_{TF}\int_F v_T = d_{TF}\int_F v_F$$

$$\int_T v_T = \sum_{F\in\mathcal{F}_T}\frac{d_{TF}}{d}\int_F v_F$$

Rule: $\quad (v_F)_{F\in\mathcal{F}_T} \longrightarrow \int_T v_T$

These rules work on any T
of any shape
(T star-shaped w.r.t. $\bar{x}_T$)

* Increase order:

$v_T \in \mathbb{P}^{k+1}(T)$: $(k \geq 0)$

(1) $\forall w \in \mathbb{P}^{k+1}(T)$:

$$\int_T \nabla v_T \cdot \nabla w = -\int_T v_T \underbrace{\Delta w}_{\mathbb{P}^{k-1}(T)} + \sum_{F\in\mathcal{F}_T} \int_F v_T \underbrace{\nabla w \cdot n_{TF}}_{\mathbb{P}^{k}(F)}$$

$$= -\int_T \Big(\pi_T^{\overbrace{k-1}^{k}} v_T\Big) \Delta w + \sum_{F\in\mathcal{F}_T} \int_T \pi_F^{k} v_T \nabla w \cdot n_{TF}$$

π_X^{ℓ} : $L^2(X)$ - orthogonal projection on $\mathbb{P}^{\ell}(X)$

Rule: $\left(\pi_T^{\overbrace{k-1}^{k}} v_T, (\pi_F^{k} v_T)\right) \longrightarrow \nabla v_T$

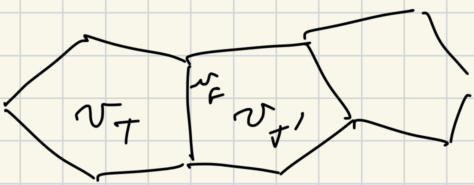
(2) $(\cancel{k \geq 1})$

$$\int_T v_T \cdot \underbrace{1}_{\in\mathbb{P}^{k-1}(T)} = \int_T \pi_T^{\overbrace{k-1}^{k}} v_T \cdot 1$$

* Local Hybrid High-Order (HHO) space:

- $\underline{U}_T^k = \{ \underline{v}_T = (v_T, (v_F)_{F \in \mathcal{F}_T}) :$

$v_T \in \mathbb{P}^k(T), \quad v_F \in \mathbb{P}^k(F) \ \forall F \}$



- Interpolant: $\underline{I}_T^k : C^0(\overline{T}) \longrightarrow \underline{U}_T^k$

$$\underline{I}_T^k v = \left(\pi_T^k v, (\pi_F^k v)_{F \in \mathcal{F}_T} \right)$$

* HHO Potential:

- Define $p_T^{k+1} : \underline{U}_T^k \longrightarrow \mathbb{P}^{k+1}(T)$ s.t.

for any $\underline{v}_T = (v_T, (v_F)_{F \in \mathcal{F}_T}) \in \underline{U}_T^k$

① $$\int_T \nabla(p_T^{k+1} \underline{v}_T) \cdot \nabla w = -\int_T v_T \Delta w + \sum_{F \in \mathcal{F}_T} \int_F v_F \nabla w \cdot n_{TF} \qquad \forall w \in \mathbb{P}^{k+1}(T)$$

② $\int_T p_T^{k+1} \underline{v}_T = \int_T v_T$

* Consistency of potential:

$$\boxed{p_T^{k+1}\left(\underline{I}_T^k q\right) = q \qquad \forall q \in \mathbb{P}^{k+1}(T)}$$

* HHO scheme ?

- Global space with BC:

$$\underline{U}_{h,0}^k = \Big\{ \underline{v}_h = \left((v_T)_{T\in\mathcal{T}_h}, (v_F)_{F\in\mathcal{F}_h} \right) : \begin{array}{ll} v_T \in \mathbb{P}^k(T) & \forall T \in \mathcal{T}_h \\ v_F \in \mathbb{P}^k(F) & \forall F \in \mathcal{F}_h \\ v_F = 0 & \text{if } F \in \mathcal{F}_h^b \end{array} \Big\}$$

- Scheme: Find $\underline{u}_h \in \underline{U}_{h,0}^k$ s.t.

$$a_h(\underline{u}_h, \underline{v}_h) = \ell_h(\underline{v}_h) \qquad \forall \underline{v}_h \in \underline{U}_{h,0}^k$$

where: $\ell_h(\underline{v}_h) = \sum_{T\in\mathcal{T}_h} \int_T f\, v_T$

and: $a_h(\underline{u}_h, \underline{v}_h) = \sum_{T\in\mathcal{T}_h} a_T(\underline{u}_T, \underline{v}_T)$

with $a_T(\underline{u}_T, \underline{v}_T) = \cancel{\int_T \nabla p_T^{k+1} \underline{u}_T \cdot \nabla p_T^{k+1} \underline{v}_T}$

- Singularity: $\underline{v}_T = \underline{v}_T$:

$$a_T(\underline{v}_T, \underline{v}_T) = \| \nabla p_T^{k+1} \underline{v}_T \|^2_{L^2(T)}$$

Expectation:

if $\| \nabla p_T^{k+1} \underline{v}_T \|^2_{L^2(T)} = 0$ then $\underline{v}_T = (C, (C)_{F \in \mathcal{F}_T})$

False: $p_T^{k+1} : \underline{U}_T^k \longrightarrow \mathbb{P}^{k+1}(T)$

dim = depends on d, k

$\underline{U}_T^k$: dim $=$ $\dim \mathbb{P}^k(T) + \sum_{F \in \mathcal{F}_T} \dim \mathbb{P}^k(F)$

$$= \dim \mathbb{P}^k(T) + \underbrace{(\# \mathcal{F}_T)}_{\text{depends on } \# \mathcal{F}_T} \dim \mathbb{P}^k(\mathbb{R}^{d-1})$$

* Poincaré inequality:

- For $\underline{v}_h \in \underline{U}_{h,0}^k$, define
$v_h \in L^2(\Omega)$: $(v_h)_{|T} = v_T \quad \forall T$

- Bound $\| v_h \|_{L^2}$?

- Take $\tau \in H^1(\Omega)^d$, $\operatorname{div} \tau = v_h$

$$\|\tau\|_{H^1} \lesssim \|v_h\|_{L^2}$$

($a \lesssim b$ means $a \le Cb$)

$$\|v_h\|_{L^2}^2 = \int_\Omega v_h \operatorname{div} \tau$$

$$= \sum_{T \in \mathcal{T}_h} -\int_T \nabla v_T \cdot \tau + \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \int_F v_T \, \tau \cdot n_{TF}$$

$$\le \Big(\sum_T \|\nabla v_T\|_{L^2(T)}^2 \Big)^{1/2} \underbrace{\|\tau\|_{L^2(\Omega)}}_{\lesssim \|v_h\|_{L^2(\Omega)}} + T_2$$

$$T_2 = \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \int_F (v_T - v_F) \, \tau \cdot n_{TF}$$

because $\displaystyle\sum_T \sum_{F \in \mathcal{F}_T} \int_F v_F \, \tau \cdot n_{TF} = 0$

$$T_2 \le \sum_T \sum_{F \in \mathcal{F}_T} h_T^{-1/2} \|v_T - v_F\|_{L^2(F)} \underbrace{h_T^{1/2} \|\tau\|_{L^2(F)}}_{\lesssim \|\tau\|_{L^2(T)} + h_T \|\nabla \tau\|_{L^2(T)}}$$

$$\underset{CS}{\lesssim} \left(\sum_T \sum_{F \in \mathcal{F}_T} h_T^{-1} \| v_T - v_F \|_{L^2(F)}^2 \right)^{1/2}$$

$$\underbrace{\left(\sum_T \| \sigma \|_{L^2(T)}^2 + h_T^2 \| \nabla \sigma \|_{L^2(T)}^2 \right)^{1/2}}_{\lesssim \| v_h \|_{L^2(\Omega)}}$$

So $\| v_h \|_{L^2}^2 \lesssim \| \underline{v}_h \|_{1,h} \| v_h \|_{L^2(\Omega)}$

where the discrete H^1-norm on $\underline{U}_{h,0}^h$

is $\| \underline{v}_h \|_{1,h}^2 = \sum_T \| \underline{v}_T \|_{1,T}^2$

where $\| \underline{v}_T \|_{1,T}^2 = \| \nabla v_T \|_{L^2(T)}^2$

$$+ \sum_{F \in \mathcal{F}_T} h_T^{-1} \| v_T - v_F \|_{L^2(F)}^2$$

Lemma (Poincaré):

$$\|v_h\|_{L^2(\Omega)} \lesssim \|\underline{v}\|_{1,h} \quad \forall \underline{v}_h \in \underline{U}^k_{h,0}$$

- Construct a_h s.t.

$$\boxed{\|\underline{u}_h\|^2_{1,h} \gtrsim a_h(\underline{u}_h, \underline{u}_h) \gtrsim \|\underline{u}_h\|^2_{1,h}}$$

If so the HHO scheme is stabe: $\|\underline{u}_h\|_{1,h} \lesssim \|f\|_{L^2(\Omega)}$

* **Stabilisation:** we look for

$$s_T : \underline{U}^k_T \times \underline{U}^k_T \longrightarrow \mathbb{R} \quad \text{bilinear symmetric}$$

s.t. $$a_T(\underline{u}_T, \underline{v}_T) = \int_T \nabla p^{k+1}_T \underline{u}_T \cdot \nabla p^{k+1}_T \underline{v}_T + s_T(\underline{u}_T, \underline{v}_T)$$

satisfies (ST1) $\|\underline{u}_T\|^2_{1,T} \gtrsim a_T(\underline{u}_T, \underline{u}_T) \gtrsim \|\underline{u}_T\|^2_{1,T}$

Consistency: if $q, r \in \mathbb{P}^{k+1}(T)$

$$a_T(\underline{I}_T^k q, \underline{I}_T^k r) = \int_T \nabla q \cdot \nabla r + s_T(\underline{I}_T^k q, \underline{I}_T^k r)$$

We want to keep this:

(ST2)
$$s_T(\underline{I}_T^k q, \underline{v}_T) = 0 \quad \forall q \in \mathbb{P}^{k+1}(T), \ \forall \underline{v}_T \in \underline{U}_T$$

* (ST2) enforces a structure:

$$s_T(\underline{u}_T, \underline{v}_T) = s_T(\underline{u}_T - \underline{I}_T^k \underbrace{p_T^{k+1} \underline{u}_T}_{\in \mathbb{P}^{k+1}(T)}, \underline{v}_T)$$

$$= s_T(\underbrace{\underline{u}_T - \underline{I}_T^k p_T^{k+1} \underline{u}_T}_{=: (\delta_T^k \underline{u}_T, (\delta_{TF}^k \underline{u}_T)_{F \in \mathcal{F}_T})}, \underline{v}_T - \underline{I}_T^{k+1} p_T^{k+1} \underline{v}_T)$$

$$\delta_T^k \underline{v}_T = v_T - \Pi_T^k \left(p_T^{k+1} \underline{v}_T \right)$$

$$\delta_{TF}^k \underline{v}_T = v_F - \Pi_F^k \left(p_T^{k+1} \underline{v}_T \right)$$

We have

$$s_T(\underline{u}_T, \underline{v}_T) = \mathcal{S}_T\left(\left(\delta_T^k \underline{u}_T, \left(\delta_{TF}^k \underline{u}_T \right)_F \right), \left(\dots \right) \right)$$

and $\delta_T^k \underline{I}_T^k q = 0$

$\delta_{TF}^k \underline{I}_T^k q = 0 \qquad \forall q \in \mathbb{P}^{k+1}$

$\|\nabla v_T\|_{L^2}^2 + h_T^{-1} \sum \|v_T - v_F\|$

$\shortparallel$

<u>Example 1:</u> $\mathcal{S}_T$ bilinear $\sim \|\cdot\|_{1,T}$

$$s_T(\underline{u}_T, \underline{v}_T) = \int_T \nabla\left(\delta_T^k \underline{u}_T\right) \cdot \nabla\left(\delta_T^k \underline{v}_T\right)$$

$$+ \; h_T^{-1} \sum_{F \in \mathcal{F}_T} \int_F \left(\delta_T^k \underline{u}_T - \delta_{TF}^k \underline{u}_T \right)\left(\delta_T^k \underline{v}_T - \delta_{TF}^k \underline{v}_T \right)$$

$$\left(s_T(\underline{v}_T, \underline{v}_T) \gtrsim \left\| \left(\delta_T \underline{v}_T, \delta_{TF} \right) \right\|_{1,T}^2 \right)$$

Example 2: (original HHO)

$$s_T(\underline{u}_T, \underline{v}_T) = h_T^{-1} \sum_{F \in \mathcal{F}_T} \int_F \left(\delta_T^k \underline{u}_T - \delta_{TF}^k \underline{u}_T\right)\left(\delta_T^k \underline{v}_T - \delta_{TF}^k \underline{v}_T\right)$$

Example 3: (inspired by VEM)

$$s_T(\underline{u}_T, \underline{v}_T) = h_T^{-2} \int_T \delta_T^k \underline{u}_T \cdot \delta_T^k \underline{v}_T + h_T^{-1} \sum_{F \in \mathcal{F}_T} \int_F \delta_{TF}^k \underline{u}_T \cdot \delta_{TF}^k \underline{v}_T$$

* HHO scheme: Find $\underline{u}_h \in \underline{U}_{h,0}^k$

s.t. $\forall \underline{v}_h \in \underline{U}_{h,0}^k$,

$$a_h(\underline{u}_h, \underline{v}_h) = \sum_T \int_T f\, v_T$$

where $a_h(\underline{u}_h, \underline{v}_h) = \sum_{T \in \mathcal{T}_h} a_T(\underline{u}_T, \underline{v}_T)$

with $a_T(\underline{u}_T, \underline{v}_T) = \int_T \nabla p_T^{k+1} \underline{u}_T \cdot \nabla p_T^{k+1} \underline{v}_T + s_T(\underline{u}_T, \underline{v}_T)$

with S_i satisfying (ST1) and (ST2).

* <u>Error estimates</u>? (3rd Strang)

$$a_h(\underline{I}_h^k u, \underline{v}_h) = \ell_h(\underline{v}_h) - \mathcal{E}_h(u; \underline{v}_h)$$

consistency error

$-$

$$a_h(\underline{u}_h, \underline{v}_h) = \ell_h(\underline{v}_h)$$

$$a_h(\underline{I}_h^k u - \underline{u}_h, \underline{v}_h) = -\mathcal{E}_h(u; \underline{v}_h)$$

Make $\underline{v}_h = \underline{I}_h^k u - \underline{u}_h$

$$\|\underline{I}_h^k u - \underline{u}_h\|_{1,h}^2 \lesssim a_h(\underline{I}_h^k u - \underline{u}_h, \underline{v}_h)$$

$$\lesssim \underbrace{\|\mathcal{E}_h(u; \cdot)\|_{1,h,*}}_{\text{dual norm to } \|\cdot\|_{1,h}} \|\underline{v}_h\|_{1,h}$$

Basic estimate: $\|\underline{I}_h^k u - \underline{u}_h\|_{1,h} \lesssim \|\mathcal{E}_h(u; \cdot)\|_{1,h,*}$

- Evaluate $\mathcal{E}_h(u; \underline{v}_h)$:

$$\mathcal{E}_h(u;\underline{v}_h) = \underbrace{\sum_T \int_T f v_T}_{T_1} - \underbrace{a_h(\underline{I}_h^k u, \underline{v}_h)}_{T_2}$$

$$T_1 = \sum_T \int_T -\Delta u \cdot v_T$$

$$= \sum_T \int_T \nabla u \cdot \nabla v_T - \sum_T \sum_{F \in \mathcal{F}_T} \int_F \nabla u \cdot n_{TF} (v_T - v_F)$$

$$T_2 = T_{2,cons} + T_{2,stab}$$

$$T_{2,cons} = \sum_T \int_T \nabla \underbrace{p_T^{k+1} \underline{I}_T^k u}_{w \in \mathbb{P}^{k+1}(T)} \cdot \nabla p_T^{k+1} \underline{v}_T$$

$$\overset{\text{def } p_T^{k+1}\underline{v}_T}{=} \sum_T \int_T -\Delta(p^{k+1} \underline{I}_T^k u)\, v_T + \sum_T \sum_{F \in \mathcal{F}_T} \int_F \nabla(p_T^{k+1} \underline{I}_T^k u) \cdot n_{TF}\, v_F$$

IBP

$$= \sum_T \int_T \nabla\left(p_T^{k+1} I_T^k u\right) \cdot \nabla v_T$$

$$+ \sum_T \sum_{F \in \mathcal{F}_T} \int_F \nabla\left(p_T^{k+1} I_T u\right) \cdot n_{TF} \left(v_F - v_T\right)$$

So $\quad T_1 - T_{2,cons}$

$$= \overbrace{\sum_T \int_T \left(\nabla u - \nabla\left(p_T^{k+1} I_T^k u\right)\right) \cdot \nabla v_T}^{= 0}$$

$$+ \underbrace{\sum_T \sum_{F \in \mathcal{F}_T} \int_F \left(\nabla u - \nabla(p I u)\right) \cdot n_{TF} \left(v_F - v_T\right)}$$

$$\leq \sum_T \sum_{F \in \mathcal{F}_T} h_T^{1/2} \left\| \nabla u - \nabla(p I u) \right\|_{L^2(F)} h_T^{-1/2} \left\| v_F - v_T \right\|_{L^2(F)}$$

$$\leq \Big(\underbrace{\sum_T \sum_F h_T \left\| \nabla u - \nabla(p I u) \right\|_{L^2(F)}^2}_{\sim\, h^{k+1} \|u\|_{H^{k+2}(\Omega)}}\Big)^{1/2} \left\| \underline{v}_h \right\|_{1,h}$$

by approx. ppty of $p \circ I$ $\left(\sim \mathbb{P}^{k+1}(T)\right)$

Th: if $v \in H^{k+2}(\Omega)$,

$$\| \underline{I}_h^k v - \underline{u}_h \|_{1,h} \lesssim h^{k+1} |v|_{H^{k+2}(\Omega)}$$

* "SPEGEO" ?

HHO is a (high-order) finite volume method.